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Convexity Estimates for Green's Function and the First Eigenfunction of Laplace Operator

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Abstract
In this paper, we find some superharmonic functions, which relate to the convexity estimates for Green's function and the first eigenfunction of Laplace operator with homogeneous Dirichlet boundary conditions in bounded convex domains of $\mathbf{R}^n$.

Keywords Laplace operator · Green's function · The first eigenfunction · Convexity estimate · Superharmonicity

Mathematics Subject Classification (2010) 35B45 · 35E10 · 35J05 · 35J08

1 Introduction

Let Ω be a bounded convex domain in $\mathbf{R}^n$, $n \geq 2$, $x_o \in \Omega$. In this paper, we will consider the convexity estimates of solutions to two elliptic partial differential equations with

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homogeneous Dirichlet boundary conditions. The first one is Green's function of Ω with pole at x_o, which is the positive solution of the following problem:

$$\begin{cases} \Delta u = -\delta(x - x_o) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\delta(x - x_o)$ denotes the Dirac measure at the point x_o. And the second one is the first eigenvalue problem of the Laplace operator:

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where λ is the first eigenvalue of the Laplace operator.

In 1920's, Carathéodory obtained that the level sets of Green's function in a bounded convex plane domain are strictly convex by using the methods of conformal mappings. In 1957, Gabriel [7] proved that the level sets of Green's function in a bounded convex three dimensional domain are strictly convex. In 1984 and 1993, from the viewpoint of probability, Borell [2, 3] proved that Green's function u of Eq. 1.1 is quasi-concave for $n = 2$ and $\frac{1}{2-n}$-convex for $n \geq 3$, i.e. the level sets of u are convex for $n = 2$ and $u^{\frac{1}{2-n}}$ is convex for $n \geq 3$.

In 2015, Shi [16] obtained a convexity estimate for Green's function of a bounded convex domain Ω with pole at $x_o \in \Omega$, and gave the proof of its specific convexity from the viewpoint of partial differential equations themselves. He proved the following theorem:

Theorem 1.1 [16] *Let Ω be a smooth bounded convex domain in $\mathbf{R}^n$, $n \geq 2$, $u > 0$ the solution for the problem* Eq. 1.1, *and $v = e^{-\alpha u}$, $\alpha > 2\pi$ for $n = 2$ or $v = u^{\frac{1}{2-n}}$ for $n \geq 3$. If v is a strictly convex function, then the function*

$$\psi_1 = v^{2-n^2} \det D^2 v,$$

that is,

$$\psi_1 = \alpha^2 \left(\det D^2 u - \alpha \sum_{i,j=1}^{2} \frac{\partial \det D^2 u}{\partial u_{ij}} u_i u_j \right) \quad \textit{for } n = 2,$$

or

$$\psi_1 = \left(\frac{1}{n-2} \right)^n \left((-1)^n u \det D^2 u + (-1)^{n-1} \frac{n-1}{n-2} \sum_{i,j=1}^{2} \frac{\partial \det D^2 u}{\partial u_{ij}} u_i u_j \right) \quad \textit{for } n \geq 3,$$

satisfies the following elliptic differential inequality:

$$\Delta \psi_1 \leq 0 \quad \textit{mod } (\nabla \psi_1) \quad \textit{in } \Omega \backslash \{x_o\},$$

where the terms containing the gradient of ψ_1 with locally bounded coefficients are suppressed. Moreover, the function ψ_1 attains its minimum on the boundary $\partial\Omega$, and for $n \geq 3$, the following estimate

$$\psi_1 \geq \frac{n-1}{(n-2)^{n+1}} \min_{\partial\Omega} K \min_{\partial\Omega} |\nabla u|^{n+1} \tag{1.3}$$

holds, where K is the Gaussian curvature of $\partial\Omega$.

Using the convexity estimate Eq. 1.3, Shi [16] combined with the deformation methods to get a new proof that v is strictly convex in $\Omega \backslash \{x_o\}$ for $n \geq 3$. In this paper, we prove that

$\log \psi_1$ is a superharmonic function exactly, which naturally leads to the convexity estimate Eq. 1.3 from Lemma 2.1 in [16].

Theorem 1.2 *Let Ω be a smooth bounded convex domain in $\mathbf{R}^n$, $n \geq 2$, $u > 0$ the solution for the problem* Eq. 1.1, *and $v = e^{-\alpha u}$, $\alpha > 2\pi$ for $n = 2$ or $v = u^{\frac{1}{2-n}}$ for $n \geq 3$. If v is a strictly convex function, then the function*

$$\varphi_1 = \log \psi_1 = \log \left(v^{2-n^2} \det D^2 v \right)$$

satisfies the following inequality:

$$\Delta \varphi_1 \leq 0 \quad \text{in } \Omega \backslash \{x_o\}.$$

In 1976, Brascamp-Lieb [5] considered the following initial-boundary value problem for the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} &= \Delta u, \quad (x,t) \in \Omega \times (0, +\infty), \\ u &= 0, (x,t) \in \partial\Omega \times (0, +\infty), \\ u(x,0) &= u_o(x), \quad x \in \Omega, \end{cases}$$

where Ω is a bounded convex domain in $\mathbf{R}^n$, $n \geq 2$, u_o is a positive function defined in Ω, and $u_o = 0$ on $\partial\Omega$. They proved that when $\log u_o$ is a concave function, $\log u$ is also concave with respect to x for any $t > 0$. From this, they established the log-concavity of the first eigenfunction of the Laplace operator for the problem Eq. 1.2 in convex domains. Korevaar [12] and Caffarelli-Spruck [6] established a maximum principle for a two-point function to give a new proof on the log-concavity of the first eigenfunction of the Laplace operator respectively, Borell [4] proved some generalizations still via probability approach.

For the case of dimension two, Acker-Payne-Philippin [1] found the following function

$$P_1 = \frac{1}{u}[u \det D^2 u + 2u_1 u_2 u_{12} - u_{11} u_2^2 - u_{22} u_1^2]$$

which satisfies

$$\Delta P_1 = 0, \quad \operatorname{mod}(\nabla P_1) \quad \text{in } \Omega \backslash \{x \in \Omega | \Theta(x) = 0\},$$

where $\Theta(x) = 4v_{12}^2 + (v_{11} - v_{22})^2$ for $v = -\log u$. Then they obtained a new proof for the Brascamp-Lieb's result in two dimensional case. The idea of using the P function to deal with partial differential equations was originally given by Makar-Limanov. Makar-Limanov [15] considered the following torsion problem in a bounded convex plane domain Ω

$$\begin{cases} \Delta u = -2 \ \text{in} \ \ \Omega, \\ \ \ u = 0 \quad \text{on} \ \partial\Omega. \end{cases} \tag{1.4}$$

He introduced an auxiliary function

$$P_2 = 2u \det D^2 u + 2u_1 u_2 u_{12} - u_{11} u_2^2 - u_{22} u_1^2,$$

and proved that P_2 is an superharmonic function. Then he could obtain that $u^{\frac{1}{2}}$ is strictly concave.

Let u be the solution of problem Eq. 1.2, and $v = -\log u$, for the function

$$\begin{aligned} \psi_2 &= \mathrm{e}^{-(n+1)v} \det D^2 v \\ &= (-1)^n u \det D^2 u + (-1)^{n-1} \sum_{i,j=1}^{n} \frac{\partial \det D^2 u}{\partial u_{ij}} u_i u_j, \end{aligned}$$

Ma-Shi-Ye [13] proved the following theorem:

Theorem 1.3 [13] *Let Ω be a smooth, bounded convex domain in $\mathbf{R}^n$, $n \geq 2$, and $u > 0$ the first eigenfunction for the eigenvalue problem* Eq. 1.2. *If $v = -\log u$ is a strictly convex function, then the function*

$$\psi_2 = e^{-(n+1)v} \det D^2 v$$

satisfies the following elliptic differential inequality:

$$\Delta \psi_2 \leq 0 \quad mod\ (\nabla \psi_2)\ in\ \Omega,$$

where the terms containing the gradient of ψ_2 with locally bounded coefficients are suppressed. Moreover, the function ψ_2 attains its minimum on the boundary $\partial\Omega$, and the following estimate

$$\psi_2 \geq \min_{\partial\Omega} K \min_{\partial\Omega} |\nabla u|^{n+1} \tag{1.5}$$

holds, where K is the Gaussian curvature of $\partial\Omega$.

Using the convexity estimate Eq. 1.5 and combining the deformation methods they gave a new proof for the Brascamp-Lieb's result in high dimensional case. In this paper, we get that $\log \psi_2$ is superharmonic, which also leads to the convexity estimate Eq. 1.5.

Theorem 1.4 *Let Ω be a smooth bounded convex domain in $\mathbf{R}^n$, $n \geq 2$ and u the solution for the problem* Eq. 1.2. *If $v = -\log u$ is a strictly convex function, then the function*

$$\varphi_2 = \log \psi_2 = \log \left(e^{-(n+1)v} \det D^2 v \right)$$

satisfies the following inequality:

$$\Delta \varphi_2 \leq 0 \quad in\ \Omega.$$

Ma-Shi-Ye [13] also gave convexity estimates for the torsion problem Eq. 1.4. It is a generalization of Makar-Limanov's result in [15] to the higher dimensions. They introduced the auxiliary functions

$$\psi_3 = (-2)^{-n} u \det D^2 u + (-2)^{-n-1} \sum_{i,j=1}^{n} \frac{\partial \det D^2 u}{\partial u_{ij}} u_i u_j,$$

and proved a differential inequality

$$\Delta \psi_3 \leq 0 \quad \text{mod}\ (\nabla \psi_3).$$

Recently, the authors in [11] further obtained that $\psi_3^{\frac{1}{n-1}}$ is superharmonic. For the harmonic functions u with convex level sets, Ma-Zhang [14] proved that $\left(|\nabla u|^{n-3} K\right)^{\frac{1}{n-1}}$ is superharmonic, where K is the Gaussian curvature of the level sets of u.

In geometric function theory and nonlinear elasticity, the superharmonic property for the log of Jacobian determinant is important to study the diffeomorphism problem, see for example in Iwaniec-Onninen [9] and Iwaniec-Koski-Onninen [10]. In higher dimension case, Gleason-Wolff [8] studied the diffeomorphism for the gradient mapping of harmonic function u, and the superharmonicity for the $\log |\det D^2 u|$ is still the key ingredient in their proof.

This paper is organized as follows. In Section 2, we give the proof of Theorem 1.2. The main technique in the proof of Theorem 1.2 consists of regrouping terms involving third order derivatives and maximizing them in each group. In Section 3, using the same process as the proof of Theorem 1.2, we prove Theorem 1.4.

2 Proof of Theorem 1.2

Firstly, we give the following elementary lemma.

Lemma 2.1 *Let B be a $(n-1)\times(n-1)$ symmetric matrix, $n \geq 2$, $x = (x_1, \cdots, x_{n-1})$, $b = (b_1, \cdots, b_{n-1}) \in \mathbf{R}^{n-1}$, and $f(x) = xBx^T + 2bx^T$. If $B < 0$, i.e., B is negative definite, then*

$$f(x) \leq -bB^{-1}b^T.$$

Proof Since $B < 0$, the polynomial $f(x)$ has a unique maximum point. At this maximum point, there hold

$$2Bx^T + 2b^T = 0,$$

equivalently, $x^T = -B^{-1}b^T$. Hence

$$\begin{aligned} f(x) &\leq (-B^{-1}b^T)^T B(-B^{-1}b^T) + 2b(-B^{-1}b^T) \\ &= \quad bB^{-1}b^T - 2bB^{-1}b^T = -bB^{-1}b^T. \end{aligned}$$

□

Now, we prove Theorem 1.2.

Proof Let u be the solution for the problem Eq. 1.1, $v = e^{-\alpha u}$, $\alpha > 2\pi$ for $n = 2$ or $v = u^{\frac{1}{2-n}}$ for $n \geq 3$. Then v is strictly convex from our assumption and satisfies the following equation and boundary conditions.

$$\Delta v = (n-1)\frac{|\nabla v|^2}{v} \quad \text{in } \Omega \setminus \{x_o\},$$
$$v(x_o) = 0,$$

and for $n = 2$,

$$v = 1 \quad \text{on } \partial\Omega,$$

or for $n \geq 3$,

$$v(x) \to +\infty \quad \text{as } x \to \partial\Omega.$$

For

$$\varphi = \log(v^{2-n^2} \det D^2 v) = (2-n^2)\log v + \log\det D^2 v,$$

we shall show that

$$\Delta\varphi \leq 0. \tag{2.1}$$

In order to prove the inequality Eq. 2.1 at an arbitrary point $\bar{x} \in \Omega$, we will choose the suitable coordinates at $\bar{x}$, such that the matrix $D^2 v(\bar{x})$ is diagonal. If we can establish Eq. 2.1 at $\bar{x}$ under the above coordinates assumption, then go back to the original coordinates we find that Eq. 2.1 remain valid. Thus it remains to establish Eq. 2.1 under the above assumption that the matrix $D^2 v(\bar{x})$ is diagonal. Because v is strictly convex, the Hessian matrix (v_{ij}) is positive definite. Denote $\lambda_i = v_{ii}(\bar{x})$ for $1 \leq i \leq n$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\sigma_1(\lambda) = \Delta v$. Let (v^{ij}) be the inverse matrix of (v_{ij}). From now on, all the calculations will be done at the fixed point $\bar{x}$ unless otherwise specified.

Take the first and second derivative of φ, we have

$$\varphi_i = \frac{(2-n^2)v_i}{v} + \sum_{k,l=1}^{n} v^{kl} v_{kli},$$

and

$$\varphi_{ii} = \frac{(2-n^2)(vv_{ii} - v_i^2)}{v^2} + \sum_{k,l=1}^{n} v^{kl} v_{klii} - \sum_{k,l,p,q=1}^{n} v^{kq} v^{pl} v_{kli} v_{pqi}.$$

Hence

$$\Delta\varphi = \frac{\left(2-n^2\right)\left(v\Delta v - |\nabla v|^2\right)}{v^2} + \sum_{k=1}^{n} v^{kk} \Delta(v_{kk}) - \sum_{k,l,i=1}^{n} v^{kk} v^{ll} v_{kli}^2. \tag{2.2}$$

By using the equation $\Delta v = (n-1)\frac{|\nabla v|^2}{v}$, we have

$$\begin{aligned}
&(\Delta v)_{kk}\\
=\;& (n-1)\left(\frac{|\nabla v|^2}{v}\right)_{kk} = (n-1)\left(2\sum_{s=1}^{n}\frac{v_s v_{sk}}{v} - \frac{|\nabla v|^2 v_k}{v^2}\right)_k\\
=\;& (n-1)\left(2\sum_{s=1}^{n}\frac{v_{sk}v_{sk}}{v} + 2\sum_{s=1}^{n}\frac{v_s v_{skk}}{v} - 2\sum_{s=1}^{n}\frac{v_s v_k v_{sk}}{v^2} - 2\sum_{s=1}^{n}\frac{v_s v_k v_{sk}}{v^2}\right) \qquad (2.3)\\
&+(n-1)\left(-\frac{|\nabla v|^2 v_{kk}}{v^2} + 2\frac{|\nabla v|^2 v_k^2}{v^3}\right)\\
=\;& (n-1)\left(2\frac{\lambda_k^2}{v} + 2\sum_{s=1}^{n}\frac{v_s v_{skk}}{v} - 4\frac{\lambda_k v_k^2}{v^2} - \frac{\lambda_k|\nabla v|^2}{v^2} + 2\frac{|\nabla v|^2 v_k^2}{v^3}\right).
\end{aligned}$$

Therefore, from Eqs. 2.2 and 2.3, we derive that

$$\begin{aligned}
\Delta\varphi =\;& -\sum_{k,i=1}^{n}\frac{v_{kki}^2}{\lambda_k^2} - \sum_{\substack{1\le k,s,i\le n\\ k\ne s}}\frac{v_{ksi}^2}{\lambda_k\lambda_s} + 2(n-1)\sum_{k,i=1}^{n}\frac{v_i}{v}\frac{v_{kki}}{\lambda_k}\\
&+(-n^3+3n^2-5n+2)\frac{|\nabla v|^2}{v^2} + 2\sum_{k=1}^{n}\frac{\sigma_1}{\lambda_k}\frac{v_k^2}{v^2}\\
\le\;& -\sum_{k,i=1}^{n}\frac{v_{kki}^2}{\lambda_k^2} - 2\sum_{\substack{1\le k,i\le n\\ k\ne i}}\frac{v_{kki}^2}{\lambda_k\lambda_i} + 2(n-1)\sum_{k,i=1}^{n}\frac{v_i}{v}\frac{v_{kki}}{\lambda_k}\\
&+\left(-n^3+3n^2-5n+2\right)\frac{|\nabla v|^2}{v^2} + 2\sum_{k=1}^{n}\frac{\sigma_1}{\lambda_k}\frac{v_k^2}{v^2},
\end{aligned}$$

the last inequality is due to $\sum_{k\ne l,k\ne i,l\ne i}\frac{v_{kli}^2}{\lambda_k\lambda_l} \ge 0$.

We claim that $\forall\ 1 \leq i \leq n$,

$$\begin{aligned} A_i \ :&= -\sum_{k=1}^{n} \frac{v_{kki}^2}{\lambda_k^2} - 2\sum_{\substack{1\leq k\leq n\\ k\neq i}} \frac{v_{kki}^2}{\lambda_k\lambda_i} + 2(n-1)\frac{v_i}{v}\sum_{k=1}^{n}\frac{v_{kki}}{\lambda_k} \\ &\quad + \left(-n^3+3n^2-5n+2\right)\frac{v_i^2}{v^2} + 2\frac{\sigma_1}{\lambda_i}\frac{v_i^2}{v^2} \\ &\leq \ 0. \end{aligned}$$

If the claim is true, we arrive at the conclusion that

$$\Delta\varphi \leq \sum_{i=1}^{n} A_i \leq 0.$$

In the following, we shall show the claim. Taking derivative of the equation $\Delta v = (n-1)\frac{|\nabla v|^2}{v}$ with respect to x_i, we have

$$(\Delta v)_i = \big(2(n-1)\lambda_i - \sigma_1(\lambda)\big)\frac{v_i}{v},$$

that is

$$v_{nni} = \big(2(n-1)\lambda_i - \sigma_1(\lambda)\big)\frac{v_i}{v} - \sum_{k=1}^{n-1} v_{kki}. \tag{2.4}$$

Applying Eq. 2.4 to A_i , we deduce that, for $1 \leq i \leq n-1$,

$$\begin{aligned} A_i \ =&\ \left(-\frac{1}{\lambda_i^2} - \frac{1}{\lambda_n^2} - \frac{2}{\lambda_i\lambda_n}\right)v_{iii}^2 \\ &+ \sum_{\substack{1\leq k\leq n-1\\ k\neq i}}\left(-\frac{1}{\lambda_k^2} - \frac{2}{\lambda_i\lambda_k} - \frac{1}{\lambda_n^2} - \frac{2}{\lambda_i\lambda_n}\right)v_{kki}^2 \\ &-\left(\frac{1}{\lambda_n^2} + \frac{2}{\lambda_i\lambda_n}\right)\sum_{\substack{1\leq k,j\leq n-1\\ k\neq j}} v_{kki}v_{jji} \\ &+\frac{v_i}{v}\sum_{j=1}^{n-1}\left[2(n-1)\left(\frac{1}{\lambda_j} - \frac{1}{\lambda_n}\right) + 2\left(\frac{1}{\lambda_n^2} + \frac{2}{\lambda_i\lambda_n}\right)\big(2(n-1)\lambda_i - \sigma_1\big)\right]v_{jji} \\ &-\left(\frac{1}{\lambda_n^2} + \frac{2}{\lambda_i\lambda_n}\right)\big(2(n-1)\lambda_i - \sigma_1\big)^2\frac{v_i^2}{v^2} + 2(n-1)\frac{2(n-1)\lambda_i - \sigma_1}{\lambda_n}\frac{v_i^2}{v^2} \\ &+\big(-n^3+3n^2-5n+2\big)\frac{v_i^2}{v^2} + 2\frac{\sigma_1(\lambda)}{\lambda_i}\frac{v_i^2}{v^2}, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}A_n =& \sum_{j=1}^{n-1}\left(-\frac{1}{\lambda_j^2}-\frac{1}{\lambda_n^2}-\frac{2}{\lambda_j\lambda_n}\right)v_{jjn}^2-\frac{1}{\lambda_n^2}\sum_{\substack{1\le j,k\le n-1\\ k\neq j}}v_{jjn}v_{kkn}\\ &+\frac{v_n}{v}\sum_{j=1}^{n-1}\left[2\frac{(n-1)\lambda_n-\sigma_1}{\lambda_n^2}+2(n-1)\frac{1}{\lambda_j}\right]v_{jjn}\\ &-\frac{1}{\lambda_n^2}\big(2(n-1)\lambda_n-\sigma_1\big)^2\frac{v_n^2}{v^2}+2(n-1)\frac{2(n-1)\lambda_n-\sigma_1}{\lambda_n}\frac{v_n^2}{v^2}\\ &+\big(-n^3+3n^2-5n+2\big)\frac{v_n^2}{v^2}+2\frac{\sigma_1}{\lambda_n}\frac{v_n^2}{v^2}.\end{aligned} \tag{2.6}$$

Therefore, we can view A_i $(1 \le i \le n)$ as a quadratic polynomial of $x_{[i]} = (v_{11i}, v_{22i}, \cdots, v_{(n-1)(n-1)i})$.

Firstly, we prove the claim for $i = 1$ and the cases for $2 \le i \le n-1$ are the same completely. Set $x_{[1]} = (v_{111}, v_{221}, \cdots, v_{(n-1)(n-1)1})$, A_1 can be rewritten as

$$A_1 = x_{[1]}B_1x_{[1]}^T + 2b_{[1]}x_{[1]}^T + d_{[1]}.$$

Note that

$$-B_1 = E_1 + F_1,$$

where

$$E_1 = \mathrm{diag}\left\{\frac{1}{\lambda_1^2}, \frac{1}{\lambda_2^2}+\frac{2}{\lambda_1\lambda_2}, \cdots, \frac{1}{\lambda_{n-1}^2}+\frac{2}{\lambda_1\lambda_{n-1}}\right\},$$

and

$$F_1 = \left(\frac{1}{\lambda_n^2}+\frac{2}{\lambda_1\lambda_n}\right)\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}(1, 1, \cdots, 1).$$

Denote

$$C_1 = \mathrm{diag}\left\{\lambda_1, \frac{\lambda_2}{\sqrt{1+2\frac{\lambda_2}{\lambda_1}}}, \cdots, \frac{\lambda_{n-1}}{\sqrt{1+2\frac{\lambda_{n-1}}{\lambda_1}}}\right\},$$

one can obtain that

$$C_1(-B_1)C_1 = I + \nu_{[1]}^T\nu_{[1]},$$

where I is $n-1$ identity matrix and $\nu_{[1]} = \sqrt{\frac{1}{\lambda_n^2}+\frac{2}{\lambda_1\lambda_n}}(1, 1, \cdots, 1)C_1$. It is easy to see that

$$B_1 < 0.$$

By Lemma 2.1, we get

$$A_1 \le -b_{[1]}B_1^{-1}b_{[1]}^T + d_{[1]}. \tag{2.7}$$

Denote $D_1 = I + \nu_{[1]}^T\nu_{[1]}$, then

$$D_1^{-1} = I - \frac{\nu_{[1]}^T\nu_{[1]}}{1+|\nu_{[1]}|^2},$$

$$-B_1^{-1} = C_1D_1^{-1}C_1.$$

Hence

$$-b_{[1]}B_1^{-1}b_{[1]}^T = b_{[1]}C_1D_1^{-1}C_1b_{[1]}^T = b_{[1]}C_1D_1^{-1}(b_{[1]}C_1)^T. \tag{2.8}$$

On the other hand, let $g = \left(1 + 2\frac{\lambda_n}{\lambda_1}\right)\frac{2(n-1)\lambda_1 - \sigma_1}{\lambda_n^2} - \frac{n-1}{\lambda_n}$. From Eq. 2.5, we have

$$b_{[1]} = \left[g(1, \cdots, 1) + (n-1)\left(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_{n-1}}\right)\right]\frac{v_1}{v}.$$

Denote $\eta_{[1]} = \left(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_{n-1}}\right)C_1$, $\bar{g} = \frac{g}{\sqrt{\frac{1}{\lambda_n^2} + \frac{2}{\lambda_1\lambda_n}}}$, then

$$b_{[1]}C_1 = \left(\bar{g}\nu_{[1]} + (n-1)\eta_{[1]}\right)\frac{v_1}{v}. \tag{2.9}$$

Therefore, combining Eqs. 2.8 and 2.9, we deduce

$$\begin{aligned}
&-b_{[1]}B_1^{-1}b_{[1]}^T \\
&= \frac{v_1^2}{v^2}\left(\bar{g}\nu_{[1]} + (n-1)\eta_{[1]}\right)D_1^{-1}\left(\bar{g}\nu_{[1]} + (n-1)\eta_{[1]}\right)^T \\
&= \frac{v_1^2}{v^2}\left(\bar{g}^2\nu_{[1]}D_1^{-1}\nu_{[1]}^T + 2(n-1)\bar{g}\eta_{[1]}D_1^{-1}\nu_{[1]}^T + (n-1)^2\eta_{[1]}D_1^{-1}\eta_{[1]}^T\right).
\end{aligned}$$

Since $D_1^{-1} = I - \frac{\nu_{[1]}^T\nu_{[1]}}{1+|\nu_{[1]}|^2}$, one can obtain that

$$\nu_{[1]}D_1^{-1}\nu_{[1]}^T = \nu_{[1]}\left(I - \frac{\nu_{[1]}^T\nu_{[1]}}{1+|\nu_{[1]}|^2}\right)\nu_{[1]}^T = \frac{|\nu_{[1]}|^2}{1+|\nu_{[1]}|^2},$$

$$\eta_{[1]}D_1^{-1}\nu_{[1]}^T = \eta_{[1]}\left(I - \frac{\nu_{[1]}^T\nu_{[1]}}{1+|\nu_{[1]}|^2}\right)\nu_{[1]}^T = \frac{\eta_{[1]}\nu_{[1]}^T}{1+|\nu_{[1]}|^2},$$

and

$$\eta_{[1]}D_1^{-1}\eta_{[1]}^T = \eta_{[1]}\left(I - \frac{\nu_{[1]}^T\nu_{[1]}}{1+|\nu_{[1]}|^2}\right)\eta_{[1]}^T = |\eta_{[1]}|^2 - \frac{(\eta_{[1]}\nu_{[1]}^T)^2}{1+|\nu_{[1]}|^2}.$$

Then

$$\begin{aligned}
&-b_{[1]}B_1^{-1}b_{[1]}^T \\
&= \frac{v_1^2}{v^2}\left(\bar{g}^2\frac{|\nu_{[1]}|^2}{1+|\nu_{[1]}|^2} + 2(n-1)\bar{g}\frac{\eta_{[1]}\nu_{[1]}^T}{1+|\nu_{[1]}|^2} + (n-1)^2|\eta_{[1]}|^2\right. \\
&\quad \left. -(n-1)^2\frac{(\eta_{[1]}\nu_{[1]}^T)^2}{1+|\nu_{[1]}|^2}\right) \qquad (2.10) \\
&= \frac{v_1^2}{v^2}\left(\bar{g}^2 + (n-1)^2|\eta_{[1]}|^2 - \frac{1}{1+|\nu_{[1]}|^2}\left((n-1)\eta_{[1]}\nu_{[1]}^T - \bar{g}\right)^2\right).
\end{aligned}$$

Hence, from Eqs. 2.5, 2.7 and 2.10, we get

$$\begin{aligned} A_1 \leq\ & -b_{[1]}B_1^{-1}b_{[1]}^T - \Big(\frac{1}{\lambda_n^2} + \frac{2}{\lambda_1\lambda_n}\Big)\big(2(n-1)\lambda_1 - \sigma_1\big)^2\frac{v_1^2}{v^2} \\ & +2(n-1)\frac{2(n-1)\lambda_1 - \sigma_1}{\lambda_n}\frac{v_1^2}{v^2} + \big(-n^3 + 3n^2 - 5n + 2\big)\frac{v_1^2}{v^2} \\ & +2\frac{\sigma_1(\lambda)}{\lambda_1}\frac{v_1^2}{v^2} \\ =\ & -\frac{1}{1+|\nu_{[1]}|^2}\big((n-1)\eta_{[1]}\nu_{[1]}^T - \bar{g}\big)^2\frac{v_1^2}{v^2} \\ & +\Big((n-1)^2|\eta_{[1]}|^2 + (n-1)^2\frac{1}{1+2\frac{\lambda_n}{\lambda_1}} - n^3 + 3n^2 - 5n + 2 + 2\frac{\sigma_1}{\lambda_1}\Big)\frac{v_1^2}{v^2}. \end{aligned} \tag{2.11}$$

By the definitions of $\nu_{[1]}$, $\eta_{[1]}$ and $\bar{g}$, direct calculation gives

$$\begin{aligned} & \big((n-1)\eta_{[1]}\nu_{[1]}^T - \bar{g}\big)^2 \\ =\ & \frac{1}{4}\Big(1+2\frac{\lambda_n}{\lambda_1}\Big)\frac{\lambda_1^2}{\lambda_n^2}\Bigg(\sum_{j=2}^n\Big(1+2\frac{\lambda_j}{\lambda_1}\Big) - (n-1)\sum_{j=2}^n\frac{1}{1+2\frac{\lambda_j}{\lambda_1}} + (n-2)(n-3)\Bigg)^2, \end{aligned}$$

and

$$1+|\nu_{[1]}|^2 = \frac{1}{4}\frac{\lambda_1^2}{\lambda_n^2}\Big(1+2\frac{\lambda_n}{\lambda_1}\Big)\Bigg(\sum_{j=2}^n\Big(1+2\frac{\lambda_j}{\lambda_1}\Big) + \sum_{j=2}^n\frac{1}{1+2\frac{\lambda_j}{\lambda_1}} + 6 - 2n\Bigg).$$

Denote $\mu_j = 1 + 2\frac{\lambda_j}{\lambda_1}$ for $2 \leq j \leq n$, then we get

$$2\frac{\sigma_1(\lambda)}{\lambda_1} = \sum_{j=2}^n \mu_j + 3 - n, \tag{2.12}$$

$$|\eta_{[1]}|^2 = 1 + \sum_{j=2}^{n-1}\frac{1}{1+2\frac{\lambda_j}{\lambda_1}} = 1 + \sum_{j=2}^{n-1}\frac{1}{\mu_j}, \tag{2.13}$$

$$((n-1)\eta_{[1]}\nu_{[1]}^T - \bar{g})^2 \tag{2.14}$$

$$= \frac{1}{4}\Big(1+2\frac{\lambda_n}{\lambda_1}\Big)\frac{\lambda_1^2}{\lambda_n^2}\Bigg(\sum_{j=2}^n\mu_j - (n-1)\sum_{j=2}^n\frac{1}{\mu_j} + (n-2)(n-3)\Bigg)^2, \tag{2.15}$$

and

$$1+|\nu_{[1]}|^2 = \frac{1}{4}\frac{\lambda_1^2}{\lambda_n^2}\Big(1+2\frac{\lambda_n}{\lambda_1}\Big)\Bigg(\sum_{j=2}^n\mu_j + \sum_{j=2}^n\frac{1}{\mu_j} + 6 - 2n\Bigg).$$

We substitute Eqs. 2.12- 2.15 into 2.11 and obtain

$$\begin{aligned} A_1 &\le -\frac{\left(\sum_{j=2}^n \mu_j - (n-1)\sum_{j=2}^n \frac{1}{\mu_j} + (n-2)(n-3)\right)^2}{\sum_{j=2}^n \mu_j + \sum_{j=2}^n \frac{1}{\mu_j} + 6 - 2n} \frac{v_1^2}{v^2} \\ &+\left((n-1)^2 \sum_{j=2}^n \frac{1}{\mu_j} + \sum_{j=2}^n \mu_j - n^3 + 4n^2 - 8n + 6\right)\frac{v_1^2}{v^2} \\ &= \frac{n^2 R(\mu_2, \cdots, \mu_n)}{\sum_{j=2}^n \mu_j + \sum_{j=2}^n \frac{1}{\mu_j} + 6 - 2n} \frac{v_1^2}{v^2}, \end{aligned}$$

where

$$R(\mu_2, \cdots, \mu_n) = \left(\sum_{j=2}^n \frac{1}{\mu_j}\right)\left(\sum_{i=2}^n \mu_i\right) - (n-2)\sum_{j=2}^n \mu_j - (n-2)\sum_{j=2}^n \frac{1}{\mu_j} + (n-1)(n-3).$$

Therefore, in order to prove

$$A_1 \le 0,$$

we only need to prove

$$R(\mu_2, \cdots, \mu_n) \le 0.$$

For $k = 2, \cdots, n$, we have

$$\begin{aligned} \frac{\partial R}{\partial \mu_k} &= \sum_{j=2}^n \frac{1}{\mu_j} - \frac{1}{\mu_k^2}\sum_{j=2}^n \mu_j + (n-2)\frac{1}{\mu_k^2} - (n-2) \\ &= \left(\sum_{\substack{2\le j\le n \\ j\ne k}} \frac{1}{\mu_j} - (n-2)\right) + \frac{1}{\mu_k^2}\left(n - 2 - \sum_{\substack{2\le j\le n \\ j\ne k}} \mu_j\right). \end{aligned}$$

Since for all $2 \le j \le n$, $\mu_j > 1$, we have

$$\sum_{\substack{2\le j\le n \\ j\ne k}} \frac{1}{\mu_j} - (n-2) < 0, \quad n - 2 - \sum_{\substack{2\le j\le n \\ j\ne k}} \mu_j < 0, \text{ and } \frac{\partial R}{\partial \mu_k} < 0.$$

Thus

$$R(\mu_2, \cdots, \mu_n) \le R(1, \cdots, 1) = 0.$$

We finished the proof of the claim for $i = 1$.

Now we prove the claim for $i = n$. Set $x_{[n]} = (v_{11n}, v_{22n}, \cdots, v_{(n-1)(n-1)n})$, A_n can be rewritten as

$$A_n = x_{[n]}^T B_n x_{[n]} + 2b_{[n]}x_{[n]} + d_{[n]}.$$

From Eq. 2.6, one can derive that

$$B_n = \left(b_{ij}^n\right)_{1\le i,j\le n-1},$$

where

$$b_{ij}^n = -\frac{1}{\lambda_n^2} - \left(\frac{1}{\lambda_j^2} + \frac{2}{\lambda_j \lambda_n}\right)\delta_{ij}.$$

It is easy to see that

$$-B_n = E_n + F_n,$$

where

$$E_n = \mathrm{diag}\left\{\frac{1}{\lambda_1^2} + \frac{2}{\lambda_1\lambda_n}, \frac{1}{\lambda_2^2} + \frac{2}{\lambda_2\lambda_n}, \cdots, \frac{1}{\lambda_{n-1}^2} + \frac{2}{\lambda_{n-1}\lambda_n}\right\},$$

and

$$F_n = \frac{1}{\lambda_n^2}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}(1, 1, \cdots, 1).$$

Denote

$$C_n = \mathrm{diag}\left\{\frac{\lambda_1}{\sqrt{1+2\frac{\lambda_1}{\lambda_n}}}, \frac{\lambda_2}{\sqrt{1+2\frac{\lambda_2}{\lambda_n}}}, \cdots, \frac{\lambda_{n-1}}{\sqrt{1+2\frac{\lambda_{n-1}}{\lambda_n}}}\right\},$$

we have

$$C_n(-B_n)C_n = I + \nu_{[n]}^T\nu_{[n]},$$

where $\nu_{[n]} = \frac{1}{\lambda_n}(1, 1, \cdots, 1)C_n$. Denote $D_n = I + \nu_{[n]}^T\nu_{[n]}$, then

$$D_n^{-1} = I - \frac{\nu_{[n]}^T\nu_{[n]}}{1+|\nu_{[n]}|^2},$$

and

$$-B_n^{-1} = C_n D_n^{-1} C_n.$$

With Eq. 2.6, we get

$$b_{[n]} = \frac{\upsilon_n}{\upsilon}\left[\frac{(n-1)\lambda_n - \sigma_1}{\lambda_n^2}(1, \cdots, 1) + (n-1)(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_{n-1}})\right].$$

Denote $\eta_{[n]} = \left(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_{n-1}}\right)C_n$, $\tilde{g} = n - 1 - \frac{\sigma_1}{\lambda_n}$, then

$$b_{[n]}C_n = \left(\tilde{g}\nu_{[n]} + (n-1)\eta_{[n]}\right)\frac{\upsilon_n}{\upsilon}.$$

Whence, we deduce that

$$\begin{aligned}
&-b_{[n]}B_n^{-1}b_{[n]}^T\\
=\ & b_{[n]}C_nD_n^{-1}(b_{[n]}C_n)^T\\
=\ & \frac{\upsilon_n^2}{\upsilon^2}\Bigg(\tilde{g}^2\frac{|\nu_{[n]}|^2}{1+|\nu_{[n]}|^2} + 2(n-1)\tilde{g}\frac{\eta_{[n]}\nu_{[n]}^T}{1+|\nu_{[n]}|^2}\\
&+(n-1)^2|\eta_{[n]}|^2 - (n-1)^2\frac{(\eta_{[n]}\nu_{[n]}^T)^2}{1+|\nu_{[n]}|^2}\Bigg)\\
=\ & \frac{\upsilon_n^2}{\upsilon^2}\left(\tilde{g}^2 + (n-1)^2|\eta_{[n]}|^2 - \frac{1}{1+|\nu_{[n]}|^2}\left((n-1)\eta_{[n]}\nu_{[n]}^T - \tilde{g}\right)^2\right).
\end{aligned} \tag{2.16}$$

Denote $\tilde{\mu}_j = 1 + 2\frac{\lambda_j}{\lambda_n}$, for $1 \le j \le n-1$, by the definition of $\nu_{[n]}$, $\eta_{[n]}$ and $\tilde{g}$, it is easy to compute

$$2\frac{\sigma_1(\lambda)}{\lambda_n} = \sum_{j=1}^{n-1}\tilde{\mu}_j + 3 - n, \tag{2.17}$$

$$|\eta_{[n]}|^2 = \sum_{j=1}^{n-1}\frac{1}{1+2\frac{\lambda_j}{\lambda_1}} = \sum_{j=1}^{n-1}\frac{1}{\tilde{\mu}_j}, \tag{2.18}$$

$$\eta_{[n]}\nu_{[n]}^T = \frac{1}{\lambda_n}\sum_{j=1}^{n-1}\frac{\lambda_j}{1+2\frac{\lambda_j}{\lambda_n}} = \frac{n-1}{2} - \frac{1}{2}\sum_{j=1}^{n-1}\frac{1}{\tilde{\mu}_j}, \tag{2.19}$$

$$\begin{aligned}
&\big((n-1)\eta_{[n]}\nu_{[n]}^T - \tilde{g}\big)^2 \\
=&\ \left(\frac{(n-1)^2}{2} - \frac{n-1}{2}\sum_{j=1}^{n-1}\frac{1}{\tilde{\mu}_j} + \frac{3-n}{2} + \frac{1}{2}\sum_{j=1}^{n-1}\tilde{\mu}_j - (n-1)\right)^2 \\
=&\ \left(-\frac{n-1}{2}\sum_{j=1}^{n-1}\frac{1}{\tilde{\mu}_j} + \frac{1}{2}\sum_{j=1}^{n-1}\tilde{\mu}_j + \frac{(n-3)(n-2)}{2}\right)^2,
\end{aligned} \tag{2.20}$$

and

$$1+|\nu_{[n]}|^2 = 1 + \frac{1}{\lambda_n^2}\sum_{j=1}^{n-1}\frac{\lambda_j^2}{1+2\frac{\lambda_j}{\lambda_n}} = \frac{1}{4}\left(\sum_{j=1}^{n-1}\tilde{\mu}_j + \sum_{j=1}^{n-1}\frac{1}{\tilde{\mu}_j} + 6 - 2n\right). \tag{2.21}$$

By Lemma 2.1, we have

$$\begin{aligned}
A_n \le&\ -b_{[n]}B_n^{-1}b_{[n]}^T - \frac{1}{\lambda_n^2}\big(2(n-1)\lambda_n - \sigma_1\big)^2\frac{v_n^2}{v^2} \\
&+2(n-1)\frac{2(n-1)\lambda_n - \sigma_1}{\lambda_n}\frac{v_n^2}{v^2} + \big(-n^3+3n^2-5n+2\big)\frac{v_n^2}{v^2} \\
&+2\frac{\sigma_1}{\lambda_n}\frac{v_n^2}{v^2}.
\end{aligned} \tag{2.22}$$

Therefore, combing with the above Eqs. 2.16-2.22, we get

$$\begin{aligned}
A_n \le&\ -\frac{\left(\sum_{j=1}^{n-1}\tilde{\mu}_j - (n-1)\sum_{j=1}^{n-1}\frac{1}{\tilde{\mu}_j} + (n-2)(n-3)\right)^2}{\sum_{j=1}^{n-1}\tilde{\mu}_j + \sum_{j=1}^{n-1}\frac{1}{\tilde{\mu}_j} + 6 - 2n}\frac{v_n^2}{v^2} \\
&+\left((n-1)^2\sum_{j=1}^{n-1}\frac{1}{\tilde{\mu}_j} + \sum_{j=1}^{n-1}\tilde{\mu}_j - n^3 + 4n^2 - 8n + 6\right)\frac{v_n^2}{v^2} \\
=&\ \frac{n^2R(\tilde{\mu}_1,\cdots,\tilde{\mu}_{n-1})}{\sum_{j=1}^{n-1}\tilde{\mu}_j + \sum_{j=1}^{n-1}\frac{1}{\tilde{\mu}_j} + 6 - 2n}\frac{v_n^2}{v^2},
\end{aligned}$$

where

$$R(\tilde{\mu}_1,\cdots,\tilde{\mu}_{n-1}) = \left(\sum_{j=1}^{n-1}\frac{1}{\tilde{\mu}_j}\right)\left(\sum_{i=1}^{n-1}\tilde{\mu}_i\right) - (n-2)\sum_{j=1}^{n-1}\mu_j - (n-2)\sum_{j=1}^{n-1}\frac{1}{\tilde{\mu}_j} + (n-1)(n-3).$$

Analogously, we can also get $R(\tilde{\mu}_1,\cdots,\tilde{\mu}_{n-1}) \le 0$, then

$$A_n \le 0.$$

This completes the proof of Theorem 1.2. □

Remark 1 In the proof of Theorem 1.2 above, if we rewrite the formula Eq. 2.4 as

$$v_{iii} = \big(2(n-1)\lambda_i - \sigma_1(\lambda)\big)\frac{v_i}{v} - \sum_{\substack{1\le k\le n\\ k\ne i}} v_{kki},$$

and eliminate the v_{iii} term in the expression of A_i, it can be processed in the form of A_n in Eq. 2.6.

3 Proof of Theorem 1.4

Now, we give the proof of Theorem 1.4. The process of the proof is similar to that of Theorem 1.2.

Proof Suppose that $u \ge 0$ is the solution for the eigenvalue problem Eq. 1.2 with $\lambda > 0$ being the first eigenvalue. Set $v = -\log u$, then v satisfies the following problem

$$\begin{cases} \Delta v = \lambda + |\nabla v|^2 & \text{in } \Omega, \\ v(x) \to +\infty & \text{as } x \to \partial\Omega. \end{cases}$$

Let $\varphi = \log(\mathrm{e}^{-(n+1)v} \det D^2 v) = -(n+1)v + \log(\det D^2 v)$, we will prove that

$$\Delta\varphi \le 0. \tag{3.1}$$

In order to show inequality Eq. 3.1 at arbitrary point $\bar{x} \in \Omega$, we choose the suitable coordinates at $\bar{x}$, such that the matrix $D^2 v(\bar{x})$ is diagonal. Because v is strictly convex, the Hessian matrix (v_{ij}) is positive definite. Let (v^{ij}) be the inverse matrix of (v_{ij}), $(\lambda_1, \lambda_2, \ldots, \lambda_n) := (v_{11}, v_{22}, \ldots, v_{nn})(\bar{x})$. From now on, all the calculations will be done at the fixed point $\bar{x}$.

By taking first derivatives of φ, we have

$$\varphi_i = -(n+1)v_i + \sum_{k,l=1}^{n} v^{kl} v_{kli}. \tag{3.2}$$

Differentiating Eq. 3.2 once more, we get

$$\Delta\varphi = -(n+1)\Delta v + \sum_{k,l=1}^{n} v^{kl}(\Delta v)_{kl} - \sum_{k,l,p,q,i=1}^{n} v^{kp} v^{ql} v_{kli} v_{pqi}.$$

Using the equation $\Delta v = \lambda + |\nabla v|^2$, we deduce

$$\sum_{k,l=1}^{n} v^{kl}(\Delta v)_{kl} = \sum_{k=1}^{n} \frac{1}{\lambda_k}(\lambda + |\nabla v|^2)_{kk} = 2\Delta v + 2\sum_{k,s=1}^{n} v_s \frac{v_{kks}}{\lambda_k}.$$

Whence

$$\begin{aligned}\Delta\varphi &= (1-n)\Delta v + 2\sum_{k,i=1}^{n} v_i \frac{v_{kki}}{\lambda_k} - \sum_{k,l,i=1}^{n} \frac{v_{kli}^2}{\lambda_k\lambda_l} \\ &= (1-n)(\lambda+|\nabla v|^2) + 2\sum_{k,i=1}^{n} v_i \frac{v_{kki}}{\lambda_k} - \sum_{k,i=1}^{n}\frac{v_{kki}^2}{\lambda_k^2} - 2\sum_{\substack{1\le k,i\le n\\ k\neq i}} \frac{v_{kki}^2}{\lambda_k\lambda_i} \\ &\quad - \sum_{\substack{1\le k,l,i\le n\\ k\neq i, l\neq i, k\neq l}} \frac{v_{kli}^2}{\lambda_k\lambda_l} \\ &\le (1-n)|\nabla v|^2 + 2\sum_{k,i=1}^{n} v_i \frac{v_{kki}}{\lambda_k} - \sum_{k,i=1}^{n}\frac{v_{kki}^2}{\lambda_k^2} - 2\sum_{\substack{1\le k,i\le n\\ k\neq i}} \frac{v_{kki}^2}{\lambda_k\lambda_i}.\end{aligned}$$

Denote

$$A_i = (1-n)v_i^2 + 2\sum_{k=1}^{n} v_i \frac{v_{kki}}{\lambda_k} - \sum_{k=1}^{n}\frac{v_{kki}^2}{\lambda_k^2} - 2\sum_{\substack{1\le k\le n\\ k\neq i}} \frac{v_{kki}^2}{\lambda_k\lambda_i}, \text{ for } 1\le i\le n.$$

We will claim that $\forall\ 1\le i\le n$,

$$A_i \le 0.$$

From this claim, we have

$$\Delta\varphi \le \sum_{i=1}^{n} A_i \le 0,$$

and complete the proof of Theorem 1.4.

In the following, we shall show the claim. Taking derivative of the equation $\Delta v = \lambda + |\nabla v|^2$ with respect to x_i, we have

$$(\Delta v)_i = 2\sum_{s=1}^{n} v_s v_{si},$$

which implies

$$v_{nni} = 2\sum_{s=1}^{n} v_s v_{si} - \sum_{k=1}^{n-1} v_{kki}. \tag{3.3}$$

Applying Eq. 3.3 to A_i , we deduce that for $1\le i\le n-1$,

$$\begin{aligned}A_i = &-\tfrac{v_{iii}^2}{\lambda_i^2} - \sum_{\substack{1\le j\le n-1\\ j\neq i}} \Big(\tfrac{1}{\lambda_j^2}+\tfrac{2}{\lambda_i\lambda_j}\Big)v_{jji}^2 - \Big(\tfrac{1}{\lambda_n^2}+\tfrac{2}{\lambda_i\lambda_n}\Big)\Big(\sum_{j=1}^{n-1} v_{jji}\Big)^2 \\ &+4\Big(\tfrac{1}{\lambda_n^2}+\tfrac{2}{\lambda_i\lambda_n}\Big)v_i\lambda_i\sum_{j=1}^{n-1} v_{jji} + 2v_i\textstyle\sum_{j=1}^{n-1}\tfrac{v_{jji}}{\lambda_j} - 2\tfrac{v_i}{\lambda_n}\displaystyle\sum_{j=1}^{n-1} v_{jji} \\ &-4v_i^2\tfrac{\lambda_i^2}{\lambda_n^2}\Big(1+2\tfrac{\lambda_n}{\lambda_i}\Big) + 4v_i^2\tfrac{\lambda_i}{\lambda_n} + (1-n)v_i^2,\end{aligned} \tag{3.4}$$

and

$$A_n = -\sum_{j=1}^{n-1}\Big(\frac{1}{\lambda_j^2}+\frac{2}{\lambda_j\lambda_n}\Big)v_{jjn}^2-\frac{1}{\lambda_n^2}\Big(\sum_{j=1}^{n-1}v_{jjn}\Big)^2+2v_n\sum_{j=1}^{n-1}\Big(\frac{1}{\lambda_j}+\frac{1}{\lambda_n}\Big)v_{jjn} \\ +(1-n)v_n^2. \tag{3.5}$$

From Eqs. 3.4 and 3.5, we can view A_i $(1 \le i \le n)$ as a quadratic polynomial of $x_{[i]} = (v_{11i}, v_{22i}, \cdots, v_{(n-1)(n-1)i})$.

Firstly, we prove the claim for $i = 1$ and the cases for $2 \le i \le n-1$ are the same completely. Set $x_{[1]} = (v_{111}, v_{221}, \cdots, v_{(n-1)(n-1)1})$, A_1 can be rewritten as

$$A_1 = x_{[1]}B_1x_{[1]}^T + 2b_{[1]}x_{[1]}^T + d_{[1]}.$$

It is easy to check that

$$-B_1 = E_1 + F_1,$$

where

$$E_1 = \mathrm{diag}\Big\{\frac{1}{\lambda_1^2}, \frac{1}{\lambda_2^2}+\frac{2}{\lambda_1\lambda_2}, \cdots, \frac{1}{\lambda_{n-1}^2}+\frac{2}{\lambda_1\lambda_{n-1}}\Big\},$$

and

$$F_1 = \Big(\frac{1}{\lambda_n^2}+\frac{2}{\lambda_1\lambda_n}\Big)\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}(1, 1, \cdots, 1).$$

Denote

$$C_1 = \mathrm{diag}\Big\{\lambda_1, \frac{\lambda_2}{\sqrt{1+2\frac{\lambda_2}{\lambda_1}}}, \cdots, \frac{\lambda_{n-1}}{\sqrt{1+2\frac{\lambda_{n-1}}{\lambda_1}}}\Big\},$$

one can obtain that

$$C_1(-B_1)C_1 = I + \nu_{[1]}^T\nu_{[1]},$$

where $\nu_{[1]} = \sqrt{\frac{1}{\lambda_n^2}+\frac{2}{\lambda_1\lambda_n}}(1, 1, \cdots, 1)C_1$. It is easy to see that

$$B_1 < 0.$$

By Lemma 2.1, we get

$$A_1 \le -b_{[1]}B_1^{-1}b_{[1]}^T + d_{[1]}. \tag{3.6}$$

Denote $D_1 = I + \nu_{[1]}^T\nu_{[1]}$, then we get

$$D_1^{-1} = I - \frac{\nu_{[1]}^T\nu_{[1]}}{1+|\nu_{[1]}|^2},$$

and

$$-B_1^{-1} = C_1D_1^{-1}C_1,$$

whence

$$-b_{[1]}B_1^{-1}b_{[1]}^T = b_{[1]}C_1D_1^{-1}(b_{[1]}C_1)^T = b_{[1]}C_1\Big(I-\frac{\nu_{[1]}^T\nu_{[1]}}{1+|\nu_{[1]}|^2}\Big)(b_{[1]}C_1)^T. \tag{3.7}$$

From Eq. 3.4, we have

$$b_{[1]} = \Big[\Big(2\frac{\lambda_1}{\lambda_n^2}+\frac{3}{\lambda_n}\Big)(1, \cdots, 1) + \Big(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_{n-1}}\Big)\Big]v_1.$$

Let $\eta_{[1]} = \left(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_{n-1}}\right)C_1$, and $g = \frac{3+2\frac{\lambda_1}{\lambda_n}}{\sqrt{1+2\frac{\lambda_n}{\lambda_1}}}$, then

$$b_{[1]}C_1 = (g\nu_{[1]} + \eta_{[1]})v_1. \tag{3.8}$$

Combing Eq. 3.7 with Eq. 3.8, we derive

$$\begin{aligned} -b_{[1]}B_1^{-1}b_{[1]}^T &= v_1^2(g\nu_{[1]} + \eta_{[1]})\Big(I - \frac{\nu_{[1]}^T\nu_{[1]}}{1+|\nu_{[1]}|^2}\Big)(g\nu_{[1]} + \eta_{[1]})^T \\ &= v_1^2\Big(g^2\frac{|\nu_{[1]}|^2}{1+|\nu_{[1]}|^2} + 2g\frac{\eta_{[1]}\nu_{[1]}^T}{1+|\nu_{[1]}|^2} + |\eta_{[1]}|^2 - \frac{(\eta_{[1]}\nu_{[1]}^T)^2}{1+|\nu_{[1]}|^2}\Big) \\ &= v_1^2\Big(g^2 + |\eta_{[1]}|^2 - \frac{1}{1+|\nu_{[1]}|^2}\big(\eta_{[1]}\nu_{[1]}^T - g\big)^2\Big). \end{aligned} \tag{3.9}$$

By the definitions of $\nu_{[1]}$ and $\eta_{[1]}$, direct calculation gives

$$\nu_{[1]} = \sqrt{\frac{1}{\lambda_n^2} + \frac{2}{\lambda_1\lambda_n}}\Big(\lambda_1, \frac{\lambda_2}{\sqrt{1+2\frac{\lambda_2}{\lambda_n}}}, \cdots, \frac{\lambda_{n-1}}{\sqrt{1+2\frac{\lambda_{n-1}}{\lambda_n}}}\Big),$$

and

$$\eta_{[1]} = \Big(1, \frac{1}{\sqrt{1+2\frac{\lambda_2}{\lambda_n}}}, \cdots, \frac{1}{\sqrt{1+2\frac{\lambda_{n-1}}{\lambda_n}}}\Big).$$

Denote $\mu_j = 1 + 2\frac{\lambda_j}{\lambda_1}$, for $2 \le j \le n$, we deduce

$$|\eta_{[1]}|^2 = 1 + \sum_{j=2}^{n-1}\frac{1}{1+2\frac{\lambda_j}{\lambda_1}} = 1 + \sum_{j=2}^{n-1}\frac{1}{\mu_j}, \tag{3.10}$$

$$\begin{aligned} 1 + |\nu_{[1]}|^2 &= 1 + \Big(1 + 2\frac{\lambda_n}{\lambda_1}\Big)\Big(\frac{\lambda_1^2}{\lambda_n^2} + \sum_{j=2}^{n-1}\frac{1}{1+2\frac{\lambda_j}{\lambda_1}}\frac{\lambda_j^2}{\lambda_n^2}\Big) \\ &= \frac{1}{4}\frac{\lambda_1^2}{\lambda_n^2}\Big(1 + 2\frac{\lambda_n}{\lambda_1}\Big)\Big(\sum_{j=2}^{n}\mu_j + \sum_{j=2}^{n}\frac{1}{\mu_j} + 6 - 2n\Big), \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} (\eta_{[1]}\nu_{[1]}^T - g)^2 &= \Big(\sqrt{\frac{1}{\lambda_n^2} + \frac{2}{\lambda_1\lambda_n}}\Big(\lambda_1 + \sum_{j=2}^{n-1}\frac{\lambda_j}{1+2\frac{\lambda_j}{\lambda_1}}\Big) - \frac{3+2\frac{\lambda_1}{\lambda_n}}{\sqrt{1+2\frac{\lambda_n}{\lambda_1}}}\Big)^2 \\ &= \frac{1}{4}\Big(1 + 2\frac{\lambda_n}{\lambda_1}\Big)\frac{\lambda_1^2}{\lambda_n^2}\Big(-\sum_{j=2}^{n}\frac{1}{\mu_j} + n - 3\Big)^2. \end{aligned} \tag{3.12}$$

Inserting Eqs. 3.9- 3.12 into 3.6, we finally obtain

$$\begin{aligned} A_1 &\le v_1^2\Big(g^2+|\eta_{[1]}|^2-\frac{1}{1+|\nu_{[1]}|^2}\Big(\eta_{[1]}\nu_{[1]}^T-g\Big)^2\Big) \\ &\quad -4v_1^2\frac{\lambda_1^2}{\lambda_n^2}(1+2\frac{\lambda_n}{\lambda_1})+4v_1^2\frac{\lambda_1}{\lambda_n}+(1-n)v_1^2 \\ &= \Bigg[-\frac{\Big(-\sum_{j=2}^n\frac{1}{\mu_j}+n-3\Big)^2}{\sum_{j=2}^n\mu_j+\sum_{j=2}^n\frac{1}{\mu_j}+6-2n}+\sum_{j=2}^n\frac{1}{\mu_j}+2-n\Bigg]v_1^2 \\ &= \frac{v_1^2}{\sum_{j=2}^n\mu_j+\sum_{j=2}^n\frac{1}{\mu_j}+6-2n}R(\mu_2,\cdots,\mu_n), \end{aligned}$$

where

$$R(\mu_2,\cdots,\mu_n)=\Bigg(\sum_{j=2}^n\frac{1}{\mu_j}\Bigg)\Bigg(\sum_{i=2}^n\mu_i\Bigg)-(n-2)\sum_{j=2}^n\mu_j-(n-2)\sum_{j=2}^n\frac{1}{\mu_j}+(n-1)(n-3).$$

By the argument in the proof of Theorem 1.2, we can get $R(\mu_2,\cdots,\mu_n)\le 0$. Hence, we have

$$A_1\le 0.$$

Now we prove the claim for $i=n$. Set $x_{[n]}=(v_{11n},v_{22n},\cdots,v_{(n-1)(n-1)n})$, A_n can be rewritten as

$$A_n=x_{[n]}^TB_nx_{[n]}+2b_{[n]}x_{[n]}+d_{[n]}.$$

From Eq. 3.5, one can derive that

$$B_n=(b_{ij}^n)_{1\le i,j\le n-1},$$

where

$$b_{ij}^n=-\frac{1}{\lambda_n^2}-\Big(\frac{1}{\lambda_j^2}+\frac{2}{\lambda_j\lambda_n}\Big)\delta_{ij}.$$

It is easy to check that

$$-B_n=E_n+F_n,$$

where

$$E_n=\mathrm{diag}\Big\{\frac{1}{\lambda_1^2}+\frac{2}{\lambda_1\lambda_n},\frac{1}{\lambda_2^2}+\frac{2}{\lambda_2\lambda_n},\cdots,\frac{1}{\lambda_{n-1}^2}+\frac{2}{\lambda_{n-1}\lambda_n}\Big\},$$

and

$$F_n=\frac{1}{\lambda_n^2}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}(1,1,\cdots,1).$$

Denote

$$C_n=\mathrm{diag}\Big\{\frac{\lambda_1}{\sqrt{1+2\frac{\lambda_1}{\lambda_n}}},\frac{\lambda_2}{\sqrt{1+2\frac{\lambda_2}{\lambda_n}}},\cdots,\frac{\lambda_{n-1}}{\sqrt{1+2\frac{\lambda_{n-1}}{\lambda_n}}}\Big\},$$

we have

$$C_n(-B_n)C_n=I+\nu_{[n]}^T\nu_{[n]},$$

where $\nu_{[n]}=\frac{1}{\lambda_n}(1,1,\cdots,1)C_n$. It follows that

$$-B_n^{-1}=C_nD_n^{-1}C_n,$$

where $D_n = I + \nu_{[n]}^T \nu_{[n]}$ and $D_n^{-1} = I - \frac{\nu_{[n]}^T \nu_{[n]}}{1+|\nu_{[n]}|^2}$. By Lemma 2.1, we get

$$A_n \le -b_{[n]} B_n^{-1} b_{[n]}^T + d_{[n]}. \tag{3.13}$$

From Eq. 3.5, we see that

$$b_{[n]} = v_n \left[\frac{1}{\lambda_n}(1, \cdots, 1) + \left(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_{n-1}} \right) \right].$$

Denote $\eta_{[n]} = \left(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_{n-1}} \right) C_n$, then one can obtain that

$$b_{[n]} C_n = (\nu_{[n]} + \eta_{[n]}) v_n.$$

Therefore

$$\begin{aligned} - b_{[n]} B_n^{-1} b_{[n]}^T &= b_{[n]} C_n D_n^{-1} (b_{[n]} C_n)^T \\ &= v_n^2 \left(\frac{|\nu_{[n]}|^2}{1 + |\nu_{[n]}|^2} + 2 \frac{\eta_{[n]} \nu_{[n]}^T}{1 + |\nu_{[n]}|^2} + |\eta_{[n]}|^2 - \frac{(\eta_{[n]} \nu_{[n]}^T)^2}{1 + |\nu_{[n]}|^2} \right) \\ &= v_n^2 \left(1 + |\eta_{[n]}|^2 - \frac{1}{1 + |\nu_{[n]}|^2} (\eta_{[n]} \nu_{[n]}^T - 1)^2 \right). \end{aligned} \tag{3.14}$$

By the definitions of $\nu_{[n]}$ and $\eta_{[n]}$, direct calculation gives

$$\nu_{[n]} = \frac{1}{\lambda_n} \left(\frac{\lambda_1}{\sqrt{1 + 2\frac{\lambda_1}{\lambda_n}}}, \cdots, \frac{\lambda_{n-1}}{\sqrt{1 + 2\frac{\lambda_{n-1}}{\lambda_n}}} \right),$$

and

$$\eta_{[n]} = \left(\frac{1}{\sqrt{1 + 2\frac{\lambda_1}{\lambda_n}}}, \cdots, \frac{1}{\sqrt{1 + 2\frac{\lambda_{n-1}}{\lambda_n}}} \right).$$

Denote $\tilde{\mu}_j = 1 + 2\frac{\lambda_j}{\lambda_n}$ for $1 \le j \le n-1$, then we compute

$$|\eta_{[n]}|^2 = \sum_{j=1}^{n-1} \frac{1}{1 + 2\frac{\lambda_j}{\lambda_n}} = \sum_{j=1}^{n-1} \frac{1}{\tilde{\mu}_j}, \tag{3.15}$$

$$\eta_{[n]} \nu_{[n]}^T = \frac{1}{\lambda_n} \sum_{j=1}^{n-1} \frac{\lambda_j}{1 + 2\frac{\lambda_j}{\lambda_n}} = \frac{n-1}{2} - \frac{1}{2} \sum_{j=1}^{n-1} \frac{1}{\tilde{\mu}_j}, \tag{3.16}$$

and

$$1 + |\nu_{[n]}|^2 = 1 + \frac{1}{\lambda_n^2} \sum_{j=1}^{n-1} \frac{\lambda_j^2}{1 + 2\frac{\lambda_j}{\lambda_n}} = \frac{1}{4} \left(\sum_{j=1}^{n-1} \tilde{\mu}_j + \sum_{j=1}^{n-1} \frac{1}{\tilde{\mu}_j} + 6 - 2n \right). \tag{3.17}$$

Inserting Eqs. 3.14-3.17 into 3.13, we finally get

$$
\begin{aligned}
A_n &\le -b_{[n]} B_n^{-1} b_{[n]}^T + (1-n) v_n^2 \\
&= v_n^2 \Big(1 + |\eta_{[n]}|^2 - \frac{1}{1 + |\nu_{[n]}|^2} (\eta_{[n]} \nu_{[n]}^T - 1)^2 + 1 - n \Big) \\
&= \left[-\frac{\Big(-\sum_{j=1}^{n-1} \frac{1}{\tilde{\mu}_j} + n - 3 \Big)^2}{\sum_{j=1}^{n-1} \tilde{\mu}_j + \sum_{j=1}^{n-1} \frac{1}{\tilde{\mu}_j} + 6 - 2n} + \sum_{j=1}^{n-1} \frac{1}{\tilde{\mu}_j} + 2 - n \right] v_n^2 \\
&= \frac{v_n^2}{\sum_{j=1}^{n-1} \tilde{\mu_j} + \sum_{j=1}^{n-1} \frac{1}{\tilde{\mu}_j} + 6 - 2n} \tilde{R}(\tilde{\mu}_1, \cdots, \tilde{\mu}_{n-1}),
\end{aligned}
$$

where

$$
\tilde{R}(\tilde{\mu}_1, \cdots, \tilde{\mu}_{n-1}) = \left(\sum_{j=1}^{n-1} \frac{1}{\tilde{\mu_j}} \right) \left(\sum_{i=1}^{n-1} \tilde{\mu}_i \right) - (n-2) \sum_{j=1}^{n-1} \tilde{\mu}_j - (n-2) \sum_{j=1}^{n-1} \frac{1}{\tilde{\mu}_j} + (n-1)(n-3).
$$

Analogously, we can also get $\tilde{R}(\tilde{\mu}_1, \cdots, \tilde{\mu}_{n-1}) \le 0$. Therefore

$$
A_n \le 0.
$$

This completes the proof of Theorem 1.4. □

Remark 2 In the proof of Theorem 1.4 above, if we rewrite the formula Eq. 3.3 as

$$
v_{iii} = 2 \sum_{s=1}^{n} v_s v_{si} - \sum_{\substack{1 \le k \le n \\ k \ne i}} v_{kki},
$$

and eliminate the v_{iii} term in the expression of A_i, it can be processed in the form of A_n in Eq. 3.5.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of Interests The authors declare that they have no conflict of interest to this work.

References

1. Acker, A., Payne, L.E., Philippin, G.: On the convexity of level lines of the fundamental mode in the clamped membrane problem, and the existence of convex solutions in a related free boundary problem. Z. Angew. Math. Phys. **32**(6), 683–694 (1981)
2. Borell, C.: Hitting probabilities of killed Brownian motion: a study on geometric regularity. Ann. Sci. École Norm. Sup. **17**(3), 451–467 (1984)
3. Borell, C.: Geometric properties of some familiar diffusions in $\mathbf{R}^n$. Ann. Probab. **21**(1), 482–489 (1993)
4. Borell, C.: Diffusion equations and geometric inequalities. Potential Analysis **12**, 49–71 (2000)

5. Brascamp, H.J., Lieb, E.H.: On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. J. Functional Analysis **22**(4), 366–389 (1976)
6. Caffarelli, L.A., Spruck, J.: Convexity properties of solutions to some classical variational problems. Comm. Partial Differential Equations **7**(11), 1337–1379 (1982)
7. Gabriel, R.M.: A result concerning convex level surfaces of 3-dimensional harmonic functions. J. London Math. Soc. **32**, 286–294 (1957)
8. Gleason, S., Wolff, T.: Lewy's harmonic gradient maps in higher dimensions. Comm. Partial Differential Equations **16**, 1925–1968 (1991)
9. Iwaniec, T., Onninen, J.: Rado-Keneser-Choquet theorem. Bull. Lond. Math. Soc. **46**(6), 1283-1291 (2014)
10. Iwaniec, T., Koski, A., Onninen, J.: Isotropic p-harmonic systems in 2D Jacobian estimates and univalent solutions. Rev. Mat. Iberoam. **32**(1), 57–77 (2016)
11. Jia, X.H., Ma, X.-N., Shi, S.J.: Remarks on convexity estimates for solutions of the torsion problem. Sci. China Math. **65**, https://doi.org/10.1007/s11425-021-1957-7 (2022)
12. Korevaar, N.: Capillary surface convexity above convex domains. Indiana Univ. Math. J. **32**(1), 73–81 (1983)
13. Ma, X.-N., Shi, S.J., Ye, Y.: The convexity estimates for the solutions of two elliptic equations. Comm. Partial Differential Equations **37**(12), 2116–2137 (2012)
14. Ma, X.-N., Zhang, W.: Superharmonicity of curvature function for the convex level sets of harmonic functions. Calc. Var. Partial Differential Equations **60**, 141 (2021)
15. Makar-Limanov, L.G.: Solution of Dirichlet's problem for the equation $\Delta u = -1$ on a convex region. Math. Notes Acad. Sci. USSR **9**, 52–53 (1971)
16. Shi, S.J.: Convexity estimates for the Green's function. Calc. Var. Partial Differential Equation **53**(3-4), 675–688 (2015)